1. A student claims that she can tell Friendly's ice cream from Herrells ice cream. As a test, she is given ten samples of ice cream (each sample is either from Friendly's or Herrells) and asked to identify each one. She is right eight times. What is the probability that she would be right exactly eight times if she guessed randomly for each sample?

$X \sim Bin(10, \frac{1}{2})$

$P(X=8) = \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2$

$= \boxed{\frac{45}{2^{10}}}$

2. Products produced by a machine has a 3 percent defective rate. What is the probability that the first defective product occurs in the fifth item inspected?

$X \sim \text{geometric}(.03)$

$P(X=5) = (.97)^4(.03)$

3. Phan is baking cookies. He mixes 400 raisins and 600 chocolate chips into her cookie dough and ends up with 500 cookies.

(a) Find the probability that a randomly picked cookie will have three raisins in it.

- Model this as a Poisson.
- Average # of raisins per cookies is $\frac{400}{500} = .8$
- X = # of raisins in a cookie

$X \sim \text{Poisson}(.8)$

$$\mathbb{P}(X=3) = e^{-.8}\frac{(.8)^3}{3!} \approx \boxed{.0383}$$

(b) Find the probability that a randomly picked cookie will have at least one chocolate chip in it.

- Average # of chocolate chips per cookie is $\frac{600}{500} = 1.2$
- Y = # of chocolate chips in a cookie

$Y \sim \text{Poisson}(1.2)$

$$\mathbb{P}(Y \geq 1) = 1 - \mathbb{P}(Y=0) = 1 - e^{-1.2}\frac{(1.2)^0}{0!} \approx \boxed{.6988}$$

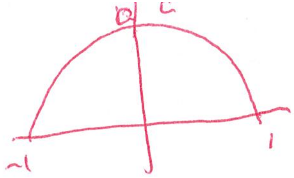
(c) Find the probability that a randomly picked cookie will have no more than two bits in it (a bit is either a raisin or a chocolate chip).

- Average # of bits per cookie is $\frac{1000}{500} = 2$
- Z = # of bits in a cookie

$Z \sim \text{Poisson}(2)$

$$\mathbb{P}(Z \leq 2) = \mathbb{P}(Z=0) + \mathbb{P}(Z=1) + \mathbb{P}(Z=2)$$
$$= e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!} \approx \boxed{.6767}$$

Note $f \geq 0$



4. Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} c\left(1 - x^6\right) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-1}^{1} c(1-x^6)\,dx = c\left[x - \frac{x^7}{7}\right]_{-1}^{1}$$

$$= c\left[\left(1 - \frac{1}{7}\right) - \left(-1 + \frac{1}{7}\right)\right] = c \cdot \frac{12}{7} \Rightarrow \boxed{c = \frac{7}{12}}$$

(b) What is the cumulative distribution function of X? Give the function, not just a sketch.

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_{-1}^{a} \frac{7}{12}(1-x^6)\,dx = \frac{7}{12}\left[x - \frac{x^7}{7}\right]_{-1}^{a}$$

$$= \boxed{\frac{7}{12}\left(a - \frac{a^7}{7} + \frac{6}{7}\right)} \quad \text{for } -1 < a < 1$$

(c) What is $\mathbb{P}(-2 < X < 0)$?

Note first since $-1 < X < 1$ that

$$\mathbb{P}(-2 < X < 0) = \mathbb{P}(X < 0) = F_X(0) = \frac{7}{12}\left[0 + \frac{6}{7}\right] = \boxed{\frac{1}{2}}$$

(d) What is $\mathbb{E}[X]$?

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-1}^{1} x \frac{7}{12}(1-x^6)\,dx = \int_{-1}^{1} \frac{7}{12}(x - x^7)\,dx$$

$$= \frac{7}{12}\left[\frac{x^2}{2} - \frac{x^8}{8}\right]_{-1}^{1} = \boxed{0}$$

(e) What is $Var(X)$?

$$\mathbb{E}X^2 = \int_{-1}^{1} x^2 \frac{7}{12}(1-x^6)\,dx = \frac{7}{12}\left[\frac{x^3}{3} - \frac{x^9}{9}\right]_{-1}^{1}$$

$$= \boxed{\frac{7}{27}}$$

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{7}{27} - 0^2 = \boxed{\frac{7}{27}}$$

5. Suppose that the time required to replace a cars windshield can be represented by an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$

   (a) What is the probability that it will take at least 3 hours to replace a windshield?

Recall from class: If $X \sim exp(\lambda)$ then $\boxed{\mathbb{P}(X > a) = e^{-\lambda a}}$

Since $X \sim exp\left(\frac{1}{2}\right)$ then

$$\mathbb{P}(X > 3) = e^{-\frac{1}{2}\cdot 3} \approx \boxed{.2231}$$

   (b) What is the probability that it will take at least 5 hours to replace a windshield given that it hasnt been finished after 2 hours?

Long way: Just compute $\mathbb{P}(X > 5 \mid X > 2)$

Short way: Recall from class that exponential R.V.'s are memoryless, thus

$$\mathbb{P}(X > 5 \mid X > 2) = \mathbb{P}(X > 2+3 \mid X > 2) = \mathbb{P}(X > 3) = e^{-\frac{3}{2}} \approx \boxed{.2231}$$

6. Suppose that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and c hours. Suppose we know that 20 percent of the repairman's jobs take less than 3 hours to finish. Find the expected amount of time a repairman need to fix a furnace.

$X \sim U(1.5, c)$ so $f_X(x) = \begin{cases} \frac{1}{c-1.5} & 1.5 < x < c \\ 0 & \text{otherwise} \end{cases}$

We know that

$$.2 = \mathbb{P}(X < 3) = \int_{1.5}^{3} \frac{1}{c-1.5}\,dx = \frac{3-1.5}{c-1.5} = \frac{1.5}{c-1.5}$$

thus

$$c - 1.5 = \frac{1.5}{.2} = 7.5 \Rightarrow \boxed{c = 7.5 + 1.5 = 9}$$

Thus $\mathbb{E}X = \frac{a+b}{2} = \frac{1.5+9}{2} = \boxed{5.25 \text{ hours}}$

Suppose that we roll 2 dice 180 times. Let E be the event that we roll two fives no more than once.

(a) Find the exact probability of E.

Success = Rolling 2 Fives, Prob of success is $\frac{1}{6^2} = \frac{1}{36}$

X = # of 2 Fives when rolling 2 dice 180 times.

$X \sim \text{Bin}(180, \frac{1}{36})$

$\mathbb{P}(E) = \mathbb{P}(X \leq 1) = \mathbb{P}(X=0) + \mathbb{P}(X=1) = \binom{180}{0}\left(\frac{35}{36}\right)^{180} + \binom{180}{1}\left(\frac{1}{36}\right)^{1}\left(\frac{35}{36}\right)^{179}$

$\approx \boxed{.0386}$

(b) Approximate $\mathbb{P}(E)$ using the normal distribution

Recall our continuity correction

$\mu = np = 180 \cdot \frac{1}{36} = 5$, $\sigma = \sqrt{180\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)} \approx 2.205$

want $\mathbb{P}(0 \leq X \leq 1) = \mathbb{P}(-.5 \leq X \leq 1.5)$

$= \mathbb{P}\left(\frac{-.5-5}{2.205} \leq Z \leq \frac{1.5-5}{2.205}\right) = (1-\Phi(1.59)) - (1-\Phi(2.49))$

$= (1-.9441) - (1-.9936) = \boxed{.0495}$

(c) Approximate $\mathbb{P}(E)$ using the Poisson distribution.

$\lambda = np = 5$

$\mathbb{P}(E) = \mathbb{P}(X=0) + \mathbb{P}(X=1)$

$\approx e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!} \approx \boxed{.0404}$

7.